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Central charges of the 6- and 19-vertex models with twisted boundary conditions

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Abstract. A new and general analytic method for calculating finite-size corrections and central charges is applied to the 6- and 19-vertex models and their related spin-$\frac{1}{2}$ and spin-1 XXZ chains with twisted boundary conditions. Nonlinear integral equations are derived from which the central charge c can be extracted in terms of Rogers dilogarithms. For twist angle ϕ, the central charge is

$$c = \frac{3S}{S+1}\left[1 - \frac{4(S+1)\phi^2}{\pi(\pi - 2S\gamma)}\right]$$

where γ is the crossing parameter or chain anisotropy and spin $S = \frac{1}{2}$ or 1. For periodic boundary conditions ($\phi = 0$) this reduces to the known results $c = 1$ and $c = \frac{3}{2}$, respectively.

1. Introduction

The critical behaviour of many two-dimensional statistical systems is described by unitary conformal field theories and classified by the central charge of the Virasoro algebra of conformal transformations [1, 2]. The central charge and scaling dimensions can be extracted from the finite-size behaviour of the eigenvalue spectrum of the transfer matrix of the statistical system or its related (1+1)-dimensional quantum spin chain [3–6].

These developments have led to a rapid growth of interest in calculating finite-size corrections in exactly solvable models. For models solvable via the Bethe ansatz, de Vega and Woynarovich [7] have given a procedure for deriving finite-size corrections using root densities. Their method has been succesfully applied to a number of models. Chief among these is the 6-vertex model [8] and the related (1+1)-dimensional spin-$\frac{1}{2}$ XXZ chain [9] due to their central role in the family of exactly solved models (see, e.g., [10–12] and references therein). Notwithstanding, the finite-size corrections to the eigenspectrum of the more general $(2S + 1)$-state vertex models and related spin-S XXZ chains (see, e.g., [13–17]) have not been calculated exactly for $S \geqslant 1$ by

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these methods. This is the case for these models despite the fact that the central charge in the antiferromagnetic region is known to be

$$c = \frac{3S}{S+1} \qquad S = \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots \tag{1.1}$$

for values of the crossing parameter or chain anisotropy γ in the range $0 \leqslant \gamma < \pi/2S$. This result has been obtained [4, 18] from the low-temperature asymptotics of the specific free energy [15–17] and is supported by direct numerical solutions of the Bethe ansatz equations for small values of S and finite chains [19–20].

At the heart of the difficulties in the higher-spin models is the nature of the complex roots of the Bethe-ansatz equations which admit a sea of $2S$-strings to describe the ground state. Although some progress has been made in calculating the asymptotic finite-size deviation to the imaginary part of the roots [21] for such ground states, the analytic calculation of the leading finite-size corrections and the central charge remains open.

Recently a new method for the *analytic* calculation of finite-size corrections, central charges and scaling dimensions of exactly solvable lattice models has been introduced [22]. This method uses integral equations, avoids the use of root densities and promises much wider applicability. In this paper we bring to fruition such an approach, as initiated in [23], to calculate the central charges of the 6- and 19-vertex models and their related spin chains.

The results for the quantum spin chains are obtained as a special limiting case of the results for the corresponding vertex model. Specifically, the energy–momentum spectrum of a quantum spin chain is given in terms of the eigenvalues of the transfer matrix $T(v)$ of the related two-dimensional lattice model by the Hamiltonian limit

$$H = \text{constant}\,(\ln T)'(v_0), \qquad P = \mathrm{i}(\ln T)(v_0) \tag{1.2}$$

where v_0 is a value of the spectral parameter at which the transfer matrix reduces to the shift operator. The advantage of working with the more general lattice models is that complex analysis can be used since then the eigenvalues $\Lambda(v)$ of $T(v)$ depend analytically on the spectral variable v.

The layout of the paper is as follows. We begin with the treatment of the 6-vertex model in section 2. The method is then extended to the 19-vertex model in section 3. Some concluding remarks are given in section 4.

2. Central charge of the 6-vertex model

In this section we calculate the central charge of the 6-vertex model and the related spin-$\frac{1}{2}$ *XXZ* chain with twisted boundary conditions. We first define the critical 6-vertex model and present the Bethe ansatz eigenvalue equations, focussing on the largest eigenvalue. Using Cauchy's theorem and other complex variable theory we recast the eigenvalue equation in the form of a nonlinear integral equation which is exact for all finite system sizes N. This equation may be regarded as a partial solution of the problem since the bulk behaviour of the eigenvalue Λ can be read off directly. Although we have not been able to solve this nonlinear integral equation analytically it is straightforward to solve it numerically. Fortunately, the $1/N$ corrections can be obtained without explicitly solving the nonlinear integral equation.

$$R^{\beta\,\delta}_{\alpha\,\gamma}(u) \;=$$

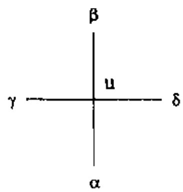

Figure 1. Labelling of the R-matrix and Boltzmann weight associated with a vertex with bond spins $\alpha, \beta, \gamma, \delta$ and spectral parameter u.

2.1. *The 6-vertex model and spin-$\frac{1}{2}$ XXZ chain*

The 6-vertex model is defined on a square lattice. Each bond or edge of the lattice carries an arrow or spin variable which takes the values $\pm\frac{1}{2}$. The Boltzmann weight of a vertex, with spins $\alpha, \beta, \gamma, \delta$ on the lower, upper, left and right bonds respectively, is given by the R-matrix $R^{\beta\ \delta}_{\alpha\ \gamma}(u)$ as shown in figure 1. The only non-zero weights are

$$\begin{aligned} R^{1/2\ 1/2}_{1/2\ 1/2}(u) &= \sinh(\lambda - u) \\ R^{1/2\ -1/2}_{-1/2\ 1/2}(u) &= \sinh\lambda \end{aligned} \tag{2.1}$$

and those related to these by the symmetries

$$R^{\beta\ \delta}_{\alpha\ \gamma}(u) = R^{-\beta\ -\delta}_{-\alpha\ -\gamma}(u) = R^{\delta\ \beta}_{\gamma\ \alpha}(u) = R^{\alpha\ \gamma}_{\beta\ \delta}(u) \tag{2.2}$$

and the crossing symmetry

$$R^{\beta\ \delta}_{\alpha\ \gamma}(u) = R^{\delta\ -\alpha}_{\gamma\ -\beta}(\lambda - u). \tag{2.3}$$

There are thus precisely six allowed vertices. The argument u is called the spectral parameter and λ is the crossing parameter. The 6-vertex model is critical for imaginary values of $\lambda = -\mathrm{i}\gamma$ with $\gamma \in [0, \pi)$. For convenience we will often work with the shifted spectral parameter

$$v = u - \lambda/2 = u + \mathrm{i}\gamma/2.$$

The 6-vertex model possesses a $U(1)$ symmetry, i.e. an invariance under rotations in spin space around the z axis. Therefore it is possible to study modified boundary conditions preserving exact integrability. We introduce a seam on the horizontal bonds linking column N to column 1. With each of these bonds we associate a local operator $\exp(2\mathrm{i}\phi S^z) = \mathrm{Diag}(\mathrm{e}^{\mathrm{i}\phi}, \mathrm{e}^{-\mathrm{i}\phi})$ changing the vertex weights in column N to

$$\mathrm{e}^{2\mathrm{i}\delta\phi} R^{\beta\ \delta}_{\alpha\ \gamma}(u). \tag{2.4}$$

In the following we will refer to ϕ as the twist angle.

The Hamiltonian limit of the row transfer matrix $T(v)$ of the 6-vertex model is taken at $v_0 = -\mathrm{i}\gamma/2$ or $u = \lambda$ as in (1.2). Normalizing by $(\mathrm{i}/2)\sin\gamma$ and subtracting the constant $(N/2)\cos\gamma$, we find the Hamiltonian of the related spin-$\frac{1}{2}$ *XXZ* chain is

$$H_{XXZ} = \sum_{j=1}^{N} [S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \cos\gamma\ S^z_j S^z_{j+1}] \tag{2.5}$$

where $S^{x,y,z}_j$ are the usual spin-$\frac{1}{2}$ operators and twisted boundary conditions have been imposed as in [24]

$$S^z_{N+1} = S^z_1 \qquad S^{\pm}_{N+1} = S^x_{N+1} \pm \mathrm{i}S^y_{N+1} = \mathrm{e}^{\pm 2\mathrm{i}\phi} S^{\pm}_1. \tag{2.6}$$

2.2. Bethe-ansatz equations

We recall that each eigenvalue $\Lambda(v)$ of the transfer matrix of the 6-vertex model satisfies the functional equation [25]

$$\Lambda(v)q(v) = \omega^{-1}\Phi(v - \mathrm{i}\gamma/2)q(v + \mathrm{i}\gamma) + \omega\Phi(v + \mathrm{i}\gamma/2)q(v - \mathrm{i}\gamma) \tag{2.7}$$

where $\Phi(v)$ and $q(v)$ are defined by

$$\Phi(v) := (\sinh v)^N \qquad q(v) := \prod_{j=1}^{m} \sinh(v - v_j) \tag{2.8}$$

and

$$\omega = \mathrm{e}^{\mathrm{i}\phi} \tag{2.9}$$

is an additional phase factor due to the twist. The unknown function $q(v)$ is the eigenvalue of an auxiliary family of transfer matrices commuting with $T(v)$. It is determined by its zeros v_j in the complex v plane.

Once the Bethe ansatz numbers v_j, i.e. the zeros of the function $q(v)$, are known the eigenvalue Λ can readily be obtained from (2.7). The numbers v_j have to satisfy a set of coupled nonlinear equations, the so-called Bethe ansatz equations,

$$p(v_j) = -1 \qquad j = 1, \cdots, m \tag{2.10}$$

where the function $p(v)$ is defined by

$$p(v) := \frac{1}{\omega^2} \frac{\Phi(v - \mathrm{i}\gamma/2)q(v + \mathrm{i}\gamma)}{\Phi(v + \mathrm{i}\gamma/2)q(v - \mathrm{i}\gamma)}. \tag{2.11}$$

For the largest eigenvalue, these zeros are distributed along the real axis.

2.3. Nonlinear integral equation

The derivation of bulk properties is usually achieved by introducing a density function ρ for the distribution of v_j on the real axis. From (2.10) a linear integral equation for ρ is then derived which can be solved by Fourier transforms. It is not easy to take into account finite-size corrections within this approach. Instead our approach is to take advantage of the fact that (2.10) renders $\Lambda(v)$ to be analytic.

We will consider systems where the finite size N is even and restrict ourselves to the study of the 'ground state' which admits $m = N/2$ real Bethe ansatz numbers. As an immediate consequence we note the following symmetry properties

$$\begin{aligned} \bar{q}(v) &= q(\bar{v}) \\ \bar{p}(v) &= 1/p(\bar{v}) \end{aligned} \tag{2.12}$$

where the bar denotes complex conjugation. Next we give some analyticity domains of the functions $\Phi(v)$, $q(v)$, $\Lambda(v)$, i.e. the strips in the complex plane where these functions are *analytic* and *non-zero* (ANZ)

$$\begin{aligned} &\Phi(v) && \text{ANZ in } 0 < \mathrm{Im}(v) < \pi \\ &q(v) && \text{ANZ in } -\pi < \mathrm{Im}(v) < 0 \\ &\Lambda(v) && \text{ANZ in } -\gamma/2 \leqslant \mathrm{Im}(v) \leqslant \gamma/2. \end{aligned} \tag{2.13}$$

We remark that the analyticity domains are by no means unique since all functions are πi-periodic. For $\Lambda(v)$ the analyticity strip follows [26] from the bulk behaviour and a corollary of Cauchy's theorem. We will take this procedure as a reliable guide for identifying analyticity domains throughout the paper. We therefore know that, although the domain of analyticity can depend on γ, the (maximal) analyticity strip of $\Lambda(v)$ is an open set containing the strip given in (2.13) and this is sufficient for our purposes. The analysis we present is applicable, in principle, to the whole regime $0 < \gamma < \pi$. For the sake of a simple presentation, however, we restrict ourselves to $0 < \gamma < \pi/2$. A simple but tedious modification also covers the regime $\pi/2 \leqslant \gamma < \pi$.

The purpose of this subsection is to derive an integral equation for the functions $\mathfrak{a}(x)$ and $\mathfrak{A}(x)$ defined by

$$\begin{aligned} \mathfrak{a}(x) &:= 1/p(x - \mathrm{i}\gamma/2) = \left[\tanh\left(\frac{\pi x}{2\gamma}\right)\right]^N a(x) \\ \mathfrak{A}(x) &:= 1 + \mathfrak{a}(x). \end{aligned} \tag{2.14}$$

These functions are central to our subsequent analysis. We have anticipated the bulk behaviour of $\mathfrak{a}(x)$ and introduced a function $a(x)$ to account for corrections. This will simplify some calculations, but it is not essential. The variable x may be regarded as real. However, sometimes it is more convenient to work with values of x in the upper half plane close to the real axis. This convention will avoid singularities which might otherwise occur.

We will often perform the Fourier transform $\mathcal{F}_k\{f\}$ of a complex function $f(v)$ which is analytic in a certain strip and decays sufficiently fast. We define the Fourier transform pair

$$\begin{aligned} \mathcal{F}_k\{f\} &:= \frac{1}{2\pi}\int f(y)\mathrm{e}^{-\mathrm{i}ky}\,\mathrm{d}y \\ f(v) &= \int_{-\infty}^{\infty} \mathcal{F}_k\{f\}\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}k \end{aligned} \tag{2.15}$$

where the integration path of the first integral has to lie in the analyticity strip and the real part of the variable of integration has to vary from $-\infty$ to ∞. By Cauchy's theorem all other details of the path are irrelevant for $\mathcal{F}_k\{f\}$.

We first apply the Fourier transform to the definition of $a(x)$

$$a(x) = \omega^2\left[\coth\left(\frac{\pi x}{2\gamma}\right)\right]^N \frac{\Phi(x)}{\Phi(x + \mathrm{i}\pi - \mathrm{i}\gamma)}\frac{q(x - \mathrm{i}3\gamma/2)}{q(x + \mathrm{i}\gamma/2 - \mathrm{i}\pi)} \tag{2.16}$$

where we have used the πi-periodicity to reduce all arguments of the functions $\Phi(v)$ and $q(v)$ to the analyticity strips (2.13). We observe that the asymptotic behaviour af these functions is given by exponentials so that the second logarithmic derivatives can be Fourier transformed. After a few manipulations this yields

$$\begin{aligned} \mathcal{F}_k[\ln a]'' = N\mathcal{F}_k\left[\ln\coth\left(\frac{\pi x}{2\gamma}\right)\right]'' + (1 - \mathrm{e}^{(\gamma-\pi)k})\mathcal{F}_k[\ln\Phi]'' \\ + (\mathrm{e}^{3\gamma k/2} - \mathrm{e}^{(\pi-\gamma/2)k})\mathcal{F}_k[\ln q]''. \end{aligned} \tag{2.17}$$

We next define an auxiliary function $h(v)$

$$h(v) := \frac{1+p(v)}{q(v)} \tag{2.18}$$

which enjoys a non-trivial ANZ property

$$h(v) \qquad \text{ANZ in } -\gamma/2 < \mathrm{Im}(v) \leqslant \gamma/2. \tag{2.19}$$

This property follows from (2.13) and the relationship with $\Lambda(v)$

$$h(v) = \frac{1}{\omega}\frac{\Lambda(v)}{\Phi(v+\mathrm{i}\gamma/2)q(v-\mathrm{i}\gamma)}. \tag{2.20}$$

To proceed, we calculate the Fourier transform of the second logarithmic derivative of h in two different ways. We choose integration paths with imaginary parts $\pm\gamma/2$ and two different representations such that the arguments of q lie in the analyticity strip (2.13),

$$\begin{aligned} h(x-\mathrm{i}\gamma/2) &= \left[\coth\left(\frac{\pi x}{2\gamma}\right)\right]^N \frac{\mathfrak{A}(x)}{q(x-\mathrm{i}\gamma/2)a(x)} \\ h(x+\mathrm{i}\gamma/2) &= \frac{\overline{\mathfrak{A}}(x)}{q(x+\mathrm{i}\gamma/2-i\pi)}. \end{aligned} \tag{2.21}$$

From these two equations we derive two formulae for $\mathcal{F}_k[\ln h]''$,

$$\begin{aligned} \mathrm{e}^{\gamma k/2}\mathcal{F}_k[\ln h]'' &= N\mathcal{F}_k\left[\ln\coth\left(\frac{\pi x}{2\gamma}\right)\right]'' - \mathrm{e}^{\gamma k/2}\mathcal{F}_k[\ln q]'' + \mathcal{F}_k[\ln\mathfrak{A}]'' - \mathcal{F}_k[\ln a]'' \\ \mathrm{e}^{-\gamma k/2}\mathcal{F}_k[\ln h]'' &= \mathcal{F}_k[\ln\overline{\mathfrak{A}}]'' - \mathrm{e}^{(\pi-\gamma/2)k}\mathcal{F}_k[\ln q]'' \end{aligned} \tag{2.22}$$

which can be equated, yielding

$$\begin{aligned} \mathcal{F}_k[\ln q]'' = {} & \frac{1}{\mathrm{e}^{(\pi+\gamma/2)k}-\mathrm{e}^{\gamma k/2}} \\ & \times\left[\mathcal{F}_k[\ln a]'' - N\mathcal{F}_k\left[\ln\coth\left(\frac{\pi x}{2\gamma}\right)\right]'' + \mathrm{e}^{\gamma k}\mathcal{F}_k[\ln\overline{\mathfrak{A}}]'' - \mathcal{F}_k[\ln\mathfrak{A}]''\right]. \end{aligned} \tag{2.23}$$

Note that this is a non-trivial identity in contrast to (2.17) which is simply a result of the definition of $a(x)$. The essential ingredient of (2.23) is the ANZ property of $\Lambda(v)$ which, for the largest eigenvalue, is equivalent to the set of Bethe-ansatz equations. Equations (2.17) and (2.23) can be solved for $\mathcal{F}_k[\ln a]''$ and $\mathcal{F}_k[\ln q]''$ in terms of $\mathcal{F}_k[\ln\mathfrak{A}]''$ and $\mathcal{F}_k[\ln\overline{\mathfrak{A}}]''$, giving

$$\mathcal{F}_k[\ln a]'' = \frac{\sinh[(\frac{1}{2}\pi-\gamma)k]}{2\cosh(\frac{1}{2}\gamma k)\sinh[\frac{1}{2}(\pi-\gamma)k]}\{\mathcal{F}_k[\ln\mathfrak{A}]'' - \mathrm{e}^{(\gamma-\epsilon)k}\mathcal{F}_k[\ln\overline{\mathfrak{A}}]''\} \tag{2.24}$$

$$\begin{aligned} \mathcal{F}_k[\ln q]'' = {} & \frac{Nk\,\mathrm{e}^{-\pi k/2}}{4\sinh(\frac{1}{2}\pi k)\cosh(\frac{1}{2}\gamma k)} \\ & + \frac{\mathrm{e}^{-(\pi+\gamma)k/2}}{4\cosh(\frac{1}{2}\gamma k)\sinh\left[\frac{1}{2}(\pi-\gamma)k\right]}\left\{\mathrm{e}^{\gamma k}\mathcal{F}_k[\ln\overline{\mathfrak{A}}]'' - \mathcal{F}_k[\ln\mathfrak{A}]''\right\}. \end{aligned} \tag{2.25}$$

Here we have used (A1.1) to evaluate the Fourier transforms of the hyperbolic functions and have introduced an infinitesimally small $\epsilon > 0$ which can be regarded as the imaginary part of the argument of x. It effectively does not change the right-hand side of (2.24) and renders the prefactor of $\mathcal{F}_k[\ln\overline{\mathfrak{A}}]''$ integrable.

We now apply the inverse Fourier transform to (2.24) where the product on the right-hand side turns into a convolution of the transforms of the individual factors,

$$[\ln \mathfrak{a}(x)]'' = \int_{-\infty}^{\infty} [F(y)\,[\ln \mathfrak{A}]''(x-y) - F(y+\mathrm{i}\epsilon-\mathrm{i}\gamma)\,[\ln\overline{\mathfrak{A}}]''(x-y)]\,\mathrm{d}y \tag{2.26}$$

where the function

$$F(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh(\frac{1}{2}\pi-\gamma)k}{2\cosh(\frac{1}{2}\gamma k)\sinh[\frac{1}{2}(\pi-\gamma)k]}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \tag{2.27}$$

satisfies the relations

$$\overline{F}(z) = F(-\bar{z}), \qquad F(-z) = F(z). \tag{2.28}$$

We integrate (2.26) twice to obtain

$$\ln \mathfrak{a}(x) = \int_{-\infty}^{\infty} \{F(y)\ln[\mathfrak{A}(x-y)] - F(y+\mathrm{i}\epsilon-\mathrm{i}\gamma)\ln[\overline{\mathfrak{A}}(x-y)]\}\,\mathrm{d}y + C + Dx. \tag{2.29}$$

The integration constants are determined by looking at the asymptotic behaviour for $x\to\infty$

$$\ln \mathfrak{a}(\infty) = \left(\int_{-\infty}^{\infty} F(y)\,\mathrm{d}y\right)\ln\mathfrak{A}(\infty) - \left(\int_{-\infty}^{\infty} F(y+\mathrm{i}\epsilon-\mathrm{i}\gamma)\,\mathrm{d}y\right)\ln\overline{\mathfrak{A}}(\infty) + C + Dx. \tag{2.30}$$

Now $\mathfrak{a}(x)$ has the same asymptotic behaviour as $\mathfrak{a}(x)$,

$$\mathfrak{a}(\pm\infty) = \omega^2 \qquad \mathfrak{A}(\pm\infty) = 1+\omega^2 \tag{2.31}$$

from which we derive

$$C = \frac{\pi\mathrm{i}\phi}{\pi-\gamma} \qquad D = 0. \tag{2.32}$$

Recalling (2.14), we then obtain the following integral equation for $\mathfrak{a}(x)$

$$\begin{aligned}\ln \mathfrak{a}(x) = N\ln\left[\tanh\left(\frac{\pi x}{2\gamma}\right)\right] \\ + \int_{-\infty}^{\infty}\left[F(y)\ln(\mathfrak{A}(x-y)) - F(y+\mathrm{i}\epsilon-\mathrm{i}\gamma)\ln\overline{\mathfrak{A}}(x-y)\right]\,\mathrm{d}y \\ + \frac{\pi\mathrm{i}\phi}{\pi-\gamma}\end{aligned} \tag{2.33}$$

which is nonlinear because of the condition $\mathfrak{A} = 1+\mathfrak{a}$. This equation is exact for all finite system sizes N.

2.4. Finite-size corrections to the largest eigenvalue

Once the solution of (2.33) is known the eigenvalue Λ can be calculated from

$$\Lambda(x-\mathrm{i}\gamma/2)=\frac{1}{\omega}\Phi(x-\mathrm{i}\gamma)\frac{q(x+\mathrm{i}\gamma/2-\mathrm{i}\pi)}{q(x-\mathrm{i}\gamma/2)}\mathfrak{A}(x). \tag{2.34}$$

The contribution given by the quotient of the q-functions is evaluated by taking the Fourier transform and then inserting (2.25)

$$\begin{aligned}\mathcal{F}_k\left[\ln\frac{q(x+\mathrm{i}\gamma/2-\mathrm{i}\pi)}{q(x-\mathrm{i}\gamma/2)}\right]'' &= (\mathrm{e}^{(\pi-\gamma/2)k}-\mathrm{e}^{\gamma k/2})\mathcal{F}_k[\ln q]''\\ &= Nk\frac{\sinh[\frac{1}{2}(\pi-\gamma)k]}{2\sinh(\frac{1}{2}\pi k)\cosh(\frac{1}{2}\gamma k)}+\frac{\mathrm{e}^{(\gamma-\epsilon)k}}{\mathrm{e}^{\gamma k}+1}\mathcal{F}_k[\ln\overline{\mathfrak{A}}]''-\frac{\mathrm{e}^{\epsilon k}}{\mathrm{e}^{\gamma k}+1}\mathcal{F}_k[\ln\mathfrak{A}]''.\end{aligned} \tag{2.35}$$

The infinitesimally small $\epsilon>0$ was introduced for the same reason as in (2.24). Applying the inverse transform, using (A1.2) and finally integrating we then obtain

$$\begin{aligned}\ln\left[\frac{q(x+\mathrm{i}\gamma/2-\mathrm{i}\pi)}{q(x-\mathrm{i}\gamma/2)}\right] &= -N\int_{-\infty}^{\infty}\frac{\sinh[(\frac{1}{2}(\pi-\gamma)k]}{2k\sinh(\frac{1}{2}\pi k)\cosh(\frac{1}{2}\gamma k)}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\\ &\quad+\frac{\mathrm{i}}{2\gamma}\int_{-\infty}^{\infty}\frac{\ln[\mathfrak{A}(x-y)]}{\sinh[(\pi/\gamma)(y-\mathrm{i}\epsilon)]}\,\mathrm{d}y+\frac{\mathrm{i}}{2\gamma}\int_{-\infty}^{\infty}\frac{\ln[\overline{\mathfrak{A}}(x-y)]}{\sinh[(\pi/\gamma)(y+\mathrm{i}\epsilon)]}\,\mathrm{d}y+C+Dx\end{aligned} \tag{2.36}$$

where C and D are constants of integration. Again from the asymptotic behaviour, we find

$$N\left(\frac{\gamma-\pi}{2}\right)\mathrm{i}=N\left(\frac{\gamma-\pi}{2}\right)\mathrm{i}-\tfrac{1}{2}\ln\mathfrak{A}(\infty)+\tfrac{1}{2}\ln\overline{\mathfrak{A}}(\infty)+C+Dx \tag{2.37}$$

where the constants are determined as

$$C=\ln\omega \qquad D=0. \tag{2.38}$$

Inserting (2.36) and (2.38) into (2.34) then gives

$$\begin{aligned}\ln\Lambda(x-\mathrm{i}\gamma/2) &= \ln\Phi(x-\mathrm{i}\gamma)-N\mathrm{i}\int_{-\infty}^{\infty}\frac{\sinh[\frac{1}{2}(\pi-\gamma)k]\sin(xk)}{2k\sinh(\frac{1}{2}\pi k)\cosh(\frac{1}{2}\gamma k)}\,\mathrm{d}k\\ &\quad+\frac{\mathrm{i}}{\gamma}\int_{-\infty}^{\infty}\frac{\operatorname{Re}\ln[\mathfrak{A}(y)]}{\sinh[(\pi/\gamma)(x-y+\mathrm{i}\epsilon)]}\,\mathrm{d}y\end{aligned} \tag{2.39}$$

where the bulk behaviour is entirely contained in the first line and the finite-size corrections are given in terms of the $\mathfrak{A}$ function alone. Again this equation is exact for all finite system sizes N.

2.5. *Analytic calculation of 1/N corrections*

To handle the asymptotic behaviour of $\Lambda(v)$ in the thermodynamic limit we observe the following scaling behaviour

$$\lim_{N\to\infty}\left[\tanh\left(\frac{\pi}{2\gamma}\right)\left(\pm\frac{\gamma}{\pi}(x+\ln N)\right)\right]^N = \exp\left(-2\mathrm{e}^{-x}\right). \tag{2.40}$$

Numerically $\mathfrak{a}(x)$ is found to scale similarly. We therefore define appropriate limiting functions in the positive and negative scaling regimes and introduce a shorthand notation for their logarithms,

$$a_\pm(x) := \lim_{N\to\infty}\mathfrak{a}\left(\pm\frac{\gamma}{\pi}(x+\ln N)\right) \qquad la_\pm(x) := \ln a_\pm(x) \tag{2.41}$$

$$A_\pm(x) := \lim_{N\to\infty}\mathfrak{A}\left(\pm\frac{\gamma}{\pi}(x+\ln N)\right) = 1 + a_\pm(x) \qquad lA_\pm(x) := \ln A_\pm(x). \tag{2.42}$$

In the scaling regimes the integral equation (2.33) simplifies to

$$la_\pm(x) = -2\mathrm{e}^{-x} + F_1 * lA_\pm - F_2 * \overline{lA}_\pm + \frac{\pi\mathrm{i}\phi}{\pi-\gamma} \tag{2.43}$$

where F_1 and F_2 are defined by

$$\begin{aligned} F_1(y) &:= \frac{\gamma}{\pi}F\left(\frac{\gamma}{\pi}y\right) \\ F_2(y) &:= \frac{\gamma}{\pi}F\left(\frac{\gamma}{\pi}y \pm \mathrm{i}(\epsilon-\gamma)\right) \end{aligned} \tag{2.44}$$

and $f * g$ denotes the convolution of two functions f and g

$$(f*g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y. \tag{2.45}$$

It will be useful to treat the functions $la_\pm$ and their complex conjugates $\overline{la}_\pm$ on an equal footing. Hence we obtain from (2.43) the following integral equation

$$\begin{pmatrix} la_\pm \\ \overline{la}_\pm \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} 2\mathrm{e}^{-x} + \begin{pmatrix} F_1 & -F_2 \\ -\overline{F_2} & \overline{F_1} \end{pmatrix} * \begin{pmatrix} lA_\pm \\ \overline{lA}_\pm \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{\pi\mathrm{i}\phi}{\pi-\gamma} \tag{2.46}$$

where the kernel

$$K := \begin{pmatrix} F_1 & -F_2 \\ -\overline{F_2} & \overline{F_1} \end{pmatrix} \tag{2.47}$$

satisfies an important symmetry property

$$K^{\mathrm{T}}(y-x) = K(x-y) \tag{2.48}$$

which will turn out to be crucial in further manipulations of (2.46).

Let us now consider the leading finite-size corrections to the eigenvalue $\Lambda(v)$. First splitting the integral in (2.39) into two parts, then substituting the variable of integration y by $\pm\frac{\gamma}{\pi}(y+\ln N)$ and finally introducing the scaling functions, we derive

$$\begin{aligned}\frac{\mathrm{i}}{\gamma}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\ln[\mathfrak{A}(y)]}{\sinh[\pi/\gamma(x-y+\mathrm{i}\epsilon)]}\,\mathrm{d}y &= \frac{\mathrm{i}}{\pi}\int_{-\ln N}^{\infty}\left[\frac{\mathrm{Re}\ln\{\mathfrak{A}[\gamma/\pi(y+\ln N)]\}}{\sinh(\pi x/\gamma-y-\ln N+\mathrm{i}\epsilon)}\right.\\ &\quad\left.+\frac{\mathrm{Re}\ln\{\mathfrak{A}[-\gamma/\pi(y+\ln N)]\}}{\sinh(\pi x/\gamma+y+\ln N+\mathrm{i}\epsilon)}\right]\mathrm{d}y\\ &\simeq -\frac{2\mathrm{i}}{\pi N}\mathrm{e}^{\pi x/\gamma}\int_{-\infty}^{\infty}\mathrm{Re}\,lA_{+}(y)\mathrm{e}^{-y}\,\mathrm{d}y+\frac{2\mathrm{i}}{\pi N}\mathrm{e}^{-\pi x/\gamma}\int_{-\infty}^{\infty}\mathrm{Re}\,lA_{-}(y)\mathrm{e}^{-y}\,\mathrm{d}y.\end{aligned} \tag{2.49}$$

Anticipating the fact that, due to symmetry, both integrals in (2.49) yield the same value and using (2.39) we obtain

$$\ln\Lambda(x-\mathrm{i}\gamma/2)\simeq -Nf(x-\mathrm{i}\gamma/2)-\frac{\pi\mathrm{i}}{6N}\sinh(\pi x/\gamma)\frac{24}{\pi^2}\int_{-\infty}^{\infty}\mathrm{Re}\,lA_{\pm}(y)\mathrm{e}^{-y}\,\mathrm{d}y. \tag{2.50}$$

In principle the nonlinear integral equation (2.43) has to be solved for $a_{\pm}$. In the previous paper [23] this equation was solved numerically. However, following [22], we are now able to calculate the integral in (2.50) without explicitly solving the integral equation.

We begin by differentiating (2.46) with respect to x

$$\begin{pmatrix} la'_{\pm} \\ \overline{la}'_{\pm}\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}2\mathrm{e}^{-x}+\begin{pmatrix}F_1 & -F_2\\ -\overline{F}_2 & \overline{F}_1\end{pmatrix}*\begin{pmatrix} lA'_{\pm}\\ \overline{lA}'_{\pm}\end{pmatrix}. \tag{2.51}$$

Multiplying (2.51) with $lA_{\pm}$, $\overline{lA}_{\pm}$ and (2.46) with $lA'_{\pm}$, $\overline{lA}'_{\pm}$, subtracting and lastly integrating we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}[la'_{\pm}(x)lA_{\pm}(x)-la_{\pm}(x)lA'_{\pm}(x)]\,\mathrm{d}x+\int_{-\infty}^{\infty}[\overline{la}'_{\pm}(x)\overline{lA}_{\pm}(x)-\overline{la}_{\pm}(x)\overline{lA}'_{\pm}(x)]\,\mathrm{d}x\\ &\qquad=2\int_{-\infty}^{\infty}\mathrm{e}^{-x}[lA_{\pm}(x)+lA'_{\pm}(x)]\,\mathrm{d}x+2\int_{-\infty}^{\infty}\mathrm{e}^{-x}[\overline{lA}_{\pm}(x)+\overline{lA}'_{\pm}(x)]\,\mathrm{d}x\\ &\qquad\quad-\frac{\pi\mathrm{i}\phi}{\pi-\gamma}\int_{-\infty}^{\infty}[lA'_{\pm}(x)-\overline{lA}'_{\pm}(x)]\,\mathrm{d}x\end{aligned} \tag{2.52}$$

where the contributions of the kernel K cancel due to the symmetry (2.48) as shown in appendix 1. After integrating by parts, the right-hand side of (2.52) is recognized essentially as the required integral in (2.50). On the other hand, the integral on the left-hand side of (2.52) can be calculated after changing the variable of integration x to a and $\overline{a}$

$$\begin{aligned}8\int_{-\infty}^{\infty}\mathrm{e}^{-x}\,\mathrm{Re}\,lA_{\pm}(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty}[la'_{\pm}lA_{\pm}-la_{\pm}lA'_{\pm}]\,\mathrm{d}x+\int_{-\infty}^{\infty}[\overline{la}'_{\pm}\overline{lA}_{\pm}-\overline{la}_{\pm}\overline{A}'_{\pm}]\,\mathrm{d}x\\ &\quad-\frac{\pi\mathrm{i}\phi}{\pi-\gamma}\ln\left(\frac{\overline{A}_{\pm}(\infty)}{A_{\pm}(\infty)}\right)\end{aligned}$$

$$= \int_{a(-\infty)}^{a(\infty)} \left[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a}\right] \mathrm{d}a + \int_{\bar{a}(-\infty)}^{\bar{a}(\infty)} \left[\frac{\ln(1+\bar{a})}{\bar{a}} - \frac{\ln \bar{a}}{1+\bar{a}}\right] \mathrm{d}\bar{a}$$

$$- \frac{\pi \mathrm{i}\phi}{\pi-\gamma} \ln\left(\frac{\overline{A}_\pm(\infty)}{A_\pm(\infty)}\right)$$

$$= 2L_+\left(\omega^2\right) + 2L_+\left(\omega^{-2}\right) - \frac{2\pi\phi^2}{\pi-\gamma}$$

$$= \frac{\pi^2}{3}\left[1 - \frac{6\phi^2}{\pi(\pi-\gamma)}\right] \tag{2.53}$$

where the asymptotics, e.g. $a_\pm(\infty) = \omega^2$ $\bar{a}_\pm(\infty) = \omega^{-2}$ $A_\pm(\infty) = 1+\omega^2$ and $\overline{A}_\pm(\infty) = 1+\omega^{-2}$, have been read off from (2.31) and (2.42). In (2.53) we introduced the dilogarithmic function

$$L_+(z) := \frac{1}{2}\int_0^z \left[\frac{\ln(1+y)}{y} - \frac{\ln y}{1+y}\right] \mathrm{d}y = L\left(\frac{z}{1+z}\right) \tag{2.54}$$

where $L(z)$ is the Rogers dilogarithm [27]. We have further used the functional equation [27]

$$L_+(z) + L_+(1/z) = \pi^2/6. \tag{2.55}$$

We are now able to give the explicit result for (2.50) in two ways

$$\ln\Lambda(x - \mathrm{i}\gamma/2) \simeq -Nf(x-\mathrm{i}\gamma/2) - \frac{\pi\mathrm{i}}{6N}\sinh\frac{\pi x}{\gamma}\left[1 - \frac{6\phi^2}{\pi(\pi-\gamma)}\right] \tag{2.56}$$

and

$$\ln\Lambda(v) \simeq -Nf(v) + \frac{\pi}{6N}\cosh\frac{\pi v}{\gamma}\left[1 - \frac{6\phi^2}{\pi(\pi-\gamma)}\right] \tag{2.57}$$

from which the central charge is easily identified as (see, e.g. [6])

$$c = 1 - \frac{6\phi^2}{\pi(\pi-\gamma)} \tag{2.58}$$

for $\phi \in (-\pi/2, \pi/2)$ in agreement with earlier numerical [24] and analytic [11] calculations.

3. Central charge of the 19-vertex model

In this section, which is the main part of the paper, we treat the 19-vertex model and the related spin-1 *XXZ* quantum chain [13]. The ideas developed in the preceding section for the 6-vertex model are directly applicable, allowing us to follow the given outline quite closely.

The 19-vertex model can be obtained from the 6-vertex model by a fusion procedure (see e.g. [17] and references therein). In the same way, higher spin vertex models and related quantum spin chains can be constructed. We will, however, restrict ourselves to the first such model and concentrate on giving a clear presentation of the method in this case. A few hints concerning the general case will be given in passing.

3.1. The 19-vertex model and spin-1 XXZ chain

The 19-vertex model is defined similarly to the 6-vertex model only now the bonds carry spin-1 variables taking the values $0, \pm 1$. The non-zero vertex weights are

$$\begin{aligned} R_{1\ 1}^{1\ 1}(u) &= \sinh(\lambda - u)\sinh(2\lambda - u) \\ R_{-1\ 1}^{1\ -1}(u) &= \sinh\lambda \sinh 2\lambda \\ R_{0\ 1}^{0\ 1}(u) &= \sinh u \sinh(\lambda - u) \\ R_{0\ 1}^{1\ 0}(u) &= \sinh 2\lambda \sinh(\lambda - u) \\ R_{0\ 0}^{0\ 0}(u) &= \sinh\lambda \sinh 2\lambda - \sinh u \sinh(\lambda - u) \end{aligned} \tag{3.1}$$

and those related to these by the symmetries (2.2) and (2.3). There are thus precisely 19 allowed vertices. The 19-vertex model is critical for imaginary values of $\lambda = -\mathrm{i}\gamma$ with $\gamma \in [0, \pi)$. There is no need to shift the spectral parameter here so we identify $v = u$. To incorporate twisted boundary conditions, a seam is introduced by local operators $\exp(2\mathrm{i}\phi S^z) = \mathrm{Diag}(e^{2\mathrm{i}\phi}, 1, e^{-2\mathrm{i}\phi})$ which changes the vertex weights in column N to

$$e^{2\mathrm{i}\delta\phi} R_{\alpha\ \gamma}^{\beta\ \delta}(u). \tag{3.2}$$

The Hamiltonian limit of the 19-vertex model row transfer matrix is taken at $v_0 = -\mathrm{i}\gamma$ or $u = \lambda$ as before. Up to normalization, this yields the spin-1 *XXZ* Hamiltonian

$$\begin{aligned} H_{XXZ} = \sum_{j=1}^{N} \{ &\tau_j - \tau_j^2 - 2(\cos\gamma - 1)(\tau_j^{\perp}\tau_j^z + \tau_j^z\tau_j^{\perp}) \\ &- 2\sin^2\gamma\ [\tau_j^z - (\tau_j^z)^2 + 2(S_j^z)^2] \} \end{aligned} \tag{3.3}$$

where $S_j^{x,y,z}$ are spin-1 operators

$$\begin{aligned} \tau_j &= S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z \\ \tau_j^z &= S_j^z S_{j+1}^z \\ \tau_j^{\perp} &= S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \end{aligned} \tag{3.4}$$

and twisted boundary conditions have been imposed as in [28]

$$S_{N+1}^z = S_1^z \qquad S_{N+1}^{\pm} = e^{\pm 2\mathrm{i}\phi} S_1^{\pm}. \tag{3.5}$$

3.2. Bethe-ansatz equations

The eigenvalues $\Lambda(v)$ of the transfer matrix of the 19-vertex model are given [17] in terms of an auxiliary function $L(v)$ by

$$\Lambda(v) = L(v)L(v + \gamma\mathrm{i}) - \sinh(v - \gamma\mathrm{i})^N \sinh(v + 2\gamma\mathrm{i})^N. \tag{3.6}$$

The function $L(v)$ is determined by a functional equation resembling (2.7),

$$L(v)q(v) = \omega^{-1}\Phi(v - \gamma\mathrm{i})q(v + \gamma\mathrm{i}) + \omega\Phi(v + \gamma\mathrm{i})q(v - \gamma\mathrm{i}) \tag{3.7}$$

where

$$\Phi(\nu) := (\sinh\nu)^N \qquad q(\nu) := \prod_{j=1}^{m} \sinh(\nu - \nu_j) \tag{3.8}$$

as before. The phase factor $\omega = e^{i\phi}$ corresponds to the twisted boundary conditions. These equations constitute the simplest example of a fusion hierarchy of functional equations. Such a hierarchy exists for each spin S and involves $2S$ auxiliary functions corresponding to $2S$ mutually commuting transfer matrices. Together with the Bethe ansatz, these equations close and therefore can be solved in principle for the eigenvalues and finite-size corrections. For spin-1, Λ and L are the eigenvalues of the two commuting families of transfer matrices.

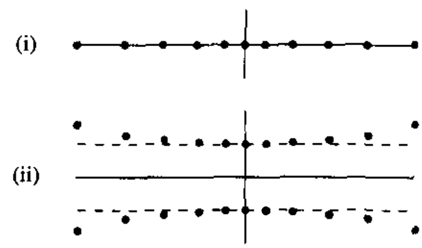

Figure 2. Schematic depiction of the distribution of Bethe-ansatz numbers in the complex ν plane for the ground state of (i) the 6-vertex model and (ii) the 19-vertex model. Notice the deviation of the Bethe-ansatz numbers from the lines Im $\nu = \pm\gamma/2$ in case (ii).

In order to render $\Lambda(\nu)$ and $L(\nu)$ analytic the numbers ν_j have to satisfy

$$p(\nu_j) = -1 \qquad j = 1, \cdots, m \tag{3.9}$$

where the function $p(\nu)$ is defined by

$$p(\nu) := \frac{1}{\omega^2} \frac{\Phi(\nu - i\gamma)q(\nu + i\gamma)}{\Phi(\nu + i\gamma)q(\nu - i\gamma)}. \tag{3.10}$$

We see from (3.6) that $\Lambda(\nu)$ can be expressed in terms of $p(\nu)$ only

$$\Lambda(\nu) = \Phi(\nu - \gamma i)\Phi(\nu + 2\gamma i) \left[\frac{1}{p(\nu)} + \frac{p(\nu + \gamma i)}{p(\nu)} + p(\nu + \gamma i)\right]. \tag{3.11}$$

The Bethe-ansatz equations (3.9) are again a set of nonlinear equations for the numbers ν_j. However, unlike for the 6-vertex model, the largest eigenvalue $\Lambda(\nu)$ admits $m = N$ zeros or Bethe-ansatz numbers ν_j grouped in 2-strings, i.e. they are distributed close to the lines $\mathrm{Im}(\nu) = \pm\gamma/2$ such that each two zeros ν_j form a complex conjugate pair. The origin of the difficulties in calculating finite-size corrections is the significant deviation of the Bethe-ansatz numbers from the lines $\mathrm{Im}(\nu) = \pm\gamma/2$ for finite systems as shown in figure 2. Although in our calculations we never need to know the precise deviations, these deviations can nevertheless be calculated within this scheme as shown in appendix 2. For periodic boundary conditions this reproduces the results of [21] specialized to spin-1.

3.3. Nonlinear integral equations

In order to accommodate the ground state we consider systems where the finite size N is even. The largest eigenvalue of the transfer matrix is then given by $N/2$ complex conjugate pairs of Bethe-ansatz numbers close to the axes $\mathrm{Im}(v) = \gamma/2$ and $\mathrm{Im}(v) = -\gamma/2$. As an immediate consequence we note that the symmetry properties (2.12) still hold. We will be particularly interested in values of the crossing parameter for which $\gamma < \pi/2$. Although the following analysis can be applied to this full range we will restrict ourselves to $\gamma < \pi/3$ for the sake of a simple presentation. Our approach is essentially based upon the ANZ property of the functions $\Phi(v)$, $q(v)$, $L(v)$, and $\Lambda(v)$ in the strips

$$\begin{aligned}
&\Phi(v) && \text{ANZ in } 0 < \mathrm{Im}(v) < \pi \\
&q(v) && \text{ANZ in } -\pi + \gamma/2 < \mathrm{Im}(v) < -\gamma/2 \\
&L(v) && \text{ANZ in } 0 \leqslant \mathrm{Im}(v) \leqslant \gamma \\
&\Lambda(v) && \text{ANZ in } -\gamma \leqslant \mathrm{Im}(v) \leqslant 0
\end{aligned} \tag{3.12}$$

where we have used the known bulk behaviour as a guide. As in the previous section these ANZ properties are exploited by setting up and solving functional relations for some suitable auxiliary functions

$$\begin{aligned}
\mathfrak{a}(x) &:= \frac{1+p(x)}{p(x)p(x-\mathrm{i}\gamma)} = \left[\tanh\left(\frac{\pi x}{2\gamma}\right)\right]^N a(x) \\
\mathfrak{b}(x) &:= \frac{1}{p(x)[1+p(x-\mathrm{i}\gamma)]} = \left[\tanh\left(\frac{\pi x}{2\gamma}\right)\right]^N b(x) \\
\mathfrak{c}(x) &:= \frac{1}{p(x-\mathrm{i}\gamma)} = \left[\tanh\left(\frac{\pi x}{2\gamma}\right)\right]^N c(x) \\
\mathfrak{A}(x) &:= 1 + \mathfrak{a}(x) \\
\mathfrak{B}(x) &:= 1 + \mathfrak{b}(x) \\
\mathfrak{C}(x) &:= 1 + \mathfrak{c}(x) = \mathfrak{A}(x)/\mathfrak{B}(x)
\end{aligned} \tag{3.13}$$

Here we have anticipated the bulk behaviour of $\mathfrak{a}(x)$, $\mathfrak{b}(x)$, $\mathfrak{c}(x)$ and have introduced functions $a(x)$, $b(x)$, $c(x)$ accounting for the corrections. The variable x may be regarded as real.

We first look at some relations among the auxiliary functions which follow simply from their definition. By inspection we obtain

$$\begin{aligned}
\frac{\bar{c}(x)}{c(x)} &= \frac{1}{\omega^4}\frac{\Phi(x+\pi\mathrm{i}-2\gamma\mathrm{i})}{\Phi(x+2\gamma\mathrm{i})}\frac{q(x+2\gamma\mathrm{i}-\pi\mathrm{i})}{q(x-2\gamma\mathrm{i})} \\
p(x) &= \frac{1}{\omega^2}\frac{\Phi(x+\pi\mathrm{i}-\gamma\mathrm{i})}{\Phi(x+\gamma\mathrm{i})}\frac{q(x+\gamma\mathrm{i}-\pi\mathrm{i})}{q(x-\gamma\mathrm{i})}
\end{aligned} \tag{3.14}$$

where we have used the πi-periodicity to reduce the arguments of $\Phi(\nu)$ and $q(\nu)$ to the analyticity strips (3.12). Taking the second logarithmic derivative and applying the Fourier transform (2.15) we get

$$\mathcal{F}_k[\ln\bar{c}]'' - \mathcal{F}_k[\ln c]'' = \frac{\sinh[(\pi/2-2\gamma)k]}{\sinh[(\pi/2-\gamma)k]}\mathcal{F}_k[\ln p]''. \tag{3.15}$$

From the relation

$$\frac{a(x)}{\bar{a}(x)} = \frac{c(x)}{\bar{c}(x)}\frac{1}{p(x)} \tag{3.16}$$

we obtain

$$\mathcal{F}_k[\ln a]'' - \mathcal{F}_k[\ln\bar{a}]'' = \mathcal{F}_k[\ln c]'' - \mathcal{F}_k[\ln\bar{c}]'' - \mathcal{F}_k[\ln p]''. \tag{3.17}$$

Similarly from

$$b(x) = \frac{c(x)}{p(x)}\frac{\mathfrak{B}(x)}{\mathfrak{A}(x)} \tag{3.18}$$

and its complex conjugate we find

$$\begin{aligned}\mathcal{F}_k[\ln b]'' &= \mathcal{F}_k[\ln c]'' - \mathcal{F}_k[\ln p]'' + \mathcal{F}_k[\ln\mathfrak{B}]'' - \mathcal{F}_k[\ln\mathfrak{A}]''\\ \mathcal{F}_k[\ln\bar{b}]'' &= \mathcal{F}_k[\ln\bar{c}]'' + \mathcal{F}_k[\ln p]'' + \mathcal{F}_k[\ln\bar{\mathfrak{B}}]'' - \mathcal{F}_k[\ln\bar{\mathfrak{A}}]''.\end{aligned} \tag{3.19}$$

We next observe that the function $a(x)$ continued to the complex plane enjoys a non-trivial ANZ property

$$a(\nu) \qquad \text{ANZ in } 0 \leqslant \mathrm{Im}(\nu) \leqslant \gamma \tag{3.20}$$

which follows from (3.12) and the relationship

$$a(\nu) = \omega^3 \frac{\Phi(\nu)\coth(\pi\nu/2\gamma)^N}{\Phi(\nu-\gamma\mathrm{i})\Phi(\nu-2\gamma\mathrm{i})}\frac{q(\nu-2\gamma\mathrm{i})}{q(\nu+\gamma\mathrm{i})}L(\nu). \tag{3.21}$$

We therefore can calculate the Fourier transform of the second logarithmic derivative of $a(\nu)$ using an integration path with imaginary part γ and employing the identity

$$a(x+\gamma\mathrm{i}) = 1/\bar{b}(x). \tag{3.22}$$

Thus we derive

$$\begin{aligned}\mathrm{e}^{-\gamma k}\mathcal{F}_k[\ln a]'' &= -\mathcal{F}_k[\ln\bar{b}]''\\ \mathrm{e}^{\gamma k}\mathcal{F}_k[\ln\bar{a}]'' &= -\mathcal{F}_k[\ln b]''\end{aligned} \tag{3.23}$$

where the second equation is obtained from the first by complex conjugation.

We apply a similar reasoning to the function

$$h(v) = \frac{1}{p(v)} + \frac{p(v+\gamma\mathrm{i})}{p(v)} + p(v+\gamma\mathrm{i}) \tag{3.24}$$

which also possesses a non-trivial ANZ property,

$$h(v) \qquad \text{ANZ in } -\gamma \leqslant \mathrm{Im}(v) \leqslant 0. \tag{3.25}$$

This is obvious from (3.12) and

$$h(v) = \Lambda(v)/\left[\Phi(v-\gamma\mathrm{i})\Phi(v+2\gamma\mathrm{i})\right]. \tag{3.26}$$

As before, we calculate the Fourier transform of the second logarithmic derivative of h in two different ways. Using the representations

$$h(x) = \frac{\overline{\mathfrak{A}}(x)}{p(x)} \tag{3.27}$$

$$h(x-\gamma\mathrm{i}) = c(x)\frac{\mathfrak{B}(x)}{b(x)} \tag{3.28}$$

we derive two formulae for $\mathcal{F}_k[\ln h]''$. Equating these gives

$$\mathcal{F}_k[\ln\overline{\mathfrak{A}}]'' - \mathcal{F}_k[\ln p]'' = \mathrm{e}^{-\gamma k}\left[\mathcal{F}_k[\ln c]'' + \mathcal{F}_k[\ln\mathfrak{B}]'' - \mathcal{F}_k[\ln b]''\right]. \tag{3.29}$$

The equations (3.15), (3.17), (3.19), (3.23) and (3.29) can be solved for $\mathcal{F}_k[\ln a]''$ etc. in terms of $\mathcal{F}_k[\ln\mathfrak{A}]''$ etc. However, we observe that in order to proceed we only need expressions for $\mathcal{F}_k[\ln ac]''$, $\mathcal{F}_k[\ln b/c]''$ and $\mathcal{F}_k[\ln p]''$

$$\begin{aligned}\mathcal{F}_k[\ln ac]'' = {} & \frac{3\mu+2\mu\nu}{(1+\mu)^2}\mathcal{F}_k[\ln\mathfrak{A}]'' - \frac{1-2\mu+2\nu}{(1+\mu)^2}\mathcal{F}_k[\ln\overline{\mathfrak{A}}]'' \\ & - \frac{\mu}{1+\mu}\mathcal{F}_k[\ln\mathfrak{B}]'' - \frac{1}{1+\mu}\mathcal{F}_k[\ln\overline{\mathfrak{B}}]''\end{aligned} \tag{3.30}$$

$$\mathcal{F}_k[\ln b/c]'' = -\frac{1}{1+\mu}\mathcal{F}_k[\ln\mathfrak{A}]'' - \frac{1}{1+\mu}\mathcal{F}_k[\ln\overline{\mathfrak{A}}]'' + \mathcal{F}_k[\ln\mathfrak{B}]'' \tag{3.31}$$

$$\mathcal{F}_k[\ln p]'' = -\frac{\mu}{1+\mu}\mathcal{F}_k[\ln\mathfrak{A}]'' + \frac{1}{1+\mu}\mathcal{F}_k[\ln\overline{\mathfrak{A}}]'' \tag{3.32}$$

where we have used the abbreviations

$$\mu := \mathrm{e}^{-\gamma k} \qquad \nu := \frac{\sinh[(\pi/2-2\gamma)k]}{\sinh[(\pi/2-\gamma)k]}. \tag{3.33}$$

Applying the inverse Fourier transform to (3.30) and (3.31) and integrating then gives

$$\begin{aligned}\ln\mathfrak{a}\mathfrak{c} &= 2N\ln\left[\tanh\left(\frac{\pi x}{2\gamma}\right)\right] + F*\ln\mathfrak{A} + G*\ln\overline{\mathfrak{A}} + H*\ln\mathfrak{B} + \overline{H}*\ln\overline{\mathfrak{B}} + \frac{2\pi\mathrm{i}\phi}{\pi-2\gamma} \\ \ln\mathfrak{b}/\mathfrak{c} &= \overline{H}*\ln\mathfrak{A} + \overline{H}*\ln\overline{\mathfrak{A}} + \ln\mathfrak{B}\end{aligned} \tag{3.34}$$

where we have introduced the functions

$$F(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{3\mu+2\mu\nu}{(1+\mu)^2}\mathrm{e}^{\mathrm{i}xk}\,\mathrm{d}k$$

$$G(x) := -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-2\mu+2\nu}{(1+\mu)^2}\mathrm{e}^{(\mathrm{i}x-\epsilon)k}\,\mathrm{d}k \tag{3.35}$$

$$H(x) := -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mu}{1+\mu}\mathrm{e}^{(\mathrm{i}x+\epsilon)k}\,\mathrm{d}k = \frac{\mathrm{i}}{2\gamma}\frac{1}{\sinh[(\pi/\gamma)(x-\mathrm{i}\epsilon)]}$$

and $\epsilon > 0$ is infinitesimally small. The integration constants in (3.34) were determined through the asymptotics

$$\mathfrak{a}(\pm\infty) = \omega^2(1+\omega^2) \qquad \mathfrak{A}(\pm\infty) = 1+\omega^2+\omega^4 \tag{3.36}$$

$$\mathfrak{b}(\pm\infty) = \frac{\omega^4}{1+\omega^2} \qquad \mathfrak{B}(\pm\infty) = \frac{1+\omega^2+\omega^4}{1+\omega^2} \tag{3.37}$$

$$\mathfrak{c}(\pm\infty) = \omega^2 \tag{3.38}$$

$$p(\pm\infty) = \frac{1}{\omega^2}. \tag{3.39}$$

In addition to (3.34) we note the subsidiary conditions $\mathfrak{A} = 1+\mathfrak{a}$ and $\mathfrak{B} = 1+\mathfrak{b}$. Now $\mathfrak{c}$ is given in terms of $\mathfrak{a}$ and $\mathfrak{b}$ by $\mathfrak{C} = 1+\mathfrak{c} = \mathfrak{A}/\mathfrak{B}$. Therefore it follows that (3.34) is a set of nonlinear integral equations for $\mathfrak{a}$ and $\mathfrak{b}$, exact for all finite system sizes N.

To summarize, we have used the ANZ properties of the fusion hierarchy in the case of spin-1 to set up nonlinear integral equations for suitably identified auxiliary functions $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$. This situation should generalize, in a straightforward way, to general spin values by introducing more auxiliary functions.

3.4. *Finite-size corrections to the eigenvalue*

Once the solution of (3.34) is known the eigenvalue $\Lambda(v)$ can be calculated from (3.26)

$$\Lambda(x-\gamma\mathrm{i}) = \Phi(x-2\gamma\mathrm{i})\Phi(x+\gamma\mathrm{i})h(x-\gamma\mathrm{i}) \tag{3.40}$$

in which h contains the finite-size corrections and is related to p and $\mathfrak{A}$ via (3.18) and (3.28)

$$h(x-\gamma\mathrm{i}) = p(x)\mathfrak{A}(x). \tag{3.41}$$

Applying the inverse transform to (3.32) and integrating we express p in terms of $\mathfrak{A}$ as

$$\ln p(x) = H * \ln\mathfrak{A} - \overline{H} * \ln\overline{\mathfrak{A}} \tag{3.42}$$

where the integration constants are zero. Then on inserting (3.42) into (3.41) and performing a contour integration,

$$\begin{aligned}\ln h(x-\gamma\mathrm{i}) &= \frac{\mathrm{i}}{2\gamma}\int_{-\infty}^{\infty}\frac{\ln[\mathfrak{A}(x-y)]}{\sinh[(\pi/\gamma)(y-\mathrm{i}\epsilon)]}\,\mathrm{d}y \\ &\quad + \frac{\mathrm{i}}{2\gamma}\int_{-\infty}^{\infty}\frac{\ln[\overline{\mathfrak{A}}(x-y)]}{\sinh[(\pi/\gamma)(y+\mathrm{i}\epsilon)]}\,\mathrm{d}y + \ln[\mathfrak{A}(x)] \\ &= \frac{\mathrm{i}}{\gamma}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\ln[\mathfrak{A}(x-y)]}{\sinh[(\pi/\gamma)(y+\mathrm{i}\epsilon)]}\,\mathrm{d}y\end{aligned} \tag{3.43}$$

we derive an expression for the eigenvalue,

$$\ln \Lambda(x-\gamma \mathrm{i}) = \ln[\Phi(x-2\gamma\mathrm{i})\Phi(x+\gamma\mathrm{i})] + \frac{\mathrm{i}}{\gamma}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\ln[\mathfrak{A}(y)]}{\sinh[(\pi/\gamma)(x-y+\mathrm{i}\epsilon)]}\,\mathrm{d}y \tag{3.44}$$

where the finite-size corrections are given exactly in terms of the $\mathfrak{A}$-function as in (2.39).

3.5. *Analytic calculation of 1/N corrections*

It is quite hopeless to look for analytic solutions of (3.34). However, it is again possible, following [22], to derive the leading finite-size corrections analytically without solving (3.34) explicitly. For this purpose we define appropriate limiting functions for $\mathfrak{a}(x)$, $\mathfrak{b}(x)$, and $\mathfrak{c}(x)$ as for $\mathfrak{a}(x)$ in (2.42). With these limiting functions and their complex conjugates, (3.34) can be put into a compact matrix notation,

$$\begin{pmatrix} la_\pm + lc_\pm \\ lb_\pm - lc_\pm \\ \overline{la}_\pm + \overline{lc}_\pm \\ \overline{lb}_\pm - \overline{lc}_\pm \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} 4\mathrm{e}^{-x} + K * \begin{pmatrix} lA_\pm \\ lB_\pm \\ \overline{lA}_\pm \\ \overline{lB}_\pm \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \left(\frac{2\pi\mathrm{i}\phi}{\pi-2\gamma}\right) \tag{3.45}$$

where the kernel K exhibits the important symmetry property (2.48). This is sufficient to apply the trick of the previous section. Multiplying the derivative of (3.45) with $(lA_\pm, lB_\pm, \overline{lA}_\pm, \overline{lB}_\pm)$ and (3.45) with $(lA'_\pm, lB'_\pm, \overline{lA}'_\pm, \overline{lB}'_\pm)$, subtracting and lastly integrating we find

$$\begin{aligned}
&\int_{-\infty}^{\infty} [(la'_\pm + lc'_\pm)lA_\pm - (la_\pm + lc_\pm)lA'_\pm]\,\mathrm{d}x \\
&\quad + \int_{-\infty}^{\infty} [(lb'_\pm - lc'_\pm)lB_\pm - (lb_\pm - lc_\pm)lB'_\pm]\,\mathrm{d}x \\
&\quad + \int_{-\infty}^{\infty} [(\overline{la}'_\pm + \overline{lc}'_\pm)\overline{lA}_\pm - (\overline{la}_\pm + \overline{lc}_\pm)\overline{lA}'_\pm]\,\mathrm{d}x \\
&\quad + \int_{-\infty}^{\infty} [(\overline{lb}'_\pm - \overline{lc}'_\pm)\overline{lB}_\pm - (\overline{lb}_\pm - \overline{lc}_\pm)\overline{lB}'_\pm]\,\mathrm{d}x \\
&= 4\int_{-\infty}^{\infty} \mathrm{e}^{-x}(lA_\pm + lA'_\pm)\,\mathrm{d}x + 4\int_{-\infty}^{\infty} \mathrm{e}^{-x}(\overline{lA}_\pm + \overline{lA}'_\pm)\,\mathrm{d}x \\
&\quad - \frac{2\pi\mathrm{i}\phi}{\pi-2\gamma}\int_{-\infty}^{\infty} [lA'_\pm(x) - \overline{lA}'_\pm(x)]\,\mathrm{d}x
\end{aligned} \tag{3.46}$$

where the contributions of K cancel due to its symmetry as shown in appendix 1. Performing an integration by parts on the right-hand side of (3.46), collecting $lc_\pm$ terms on the left-hand side and employing the relation $lA_\pm - lB_\pm = lC_\pm$ we then derive

$$16\int_{-\infty}^{\infty} e^{-x}\,\mathrm{Re}\,lA_\pm\,dx = \int_{-\infty}^{\infty}[la'_\pm lA_\pm - la_\pm lA'_\pm]\,dx + \int_{-\infty}^{\infty}[\overline{la}'_\pm \overline{lA}_\pm - \overline{la}_\pm \overline{lA}'_\pm]\,dx$$
$$+\int_{-\infty}^{\infty}[lb'_\pm lB_\pm - lb_\pm lB'_\pm]\,dx + \int_{-\infty}^{\infty}[\overline{lb}'_\pm \overline{lB}_\pm - \overline{lb}_\pm \overline{lB}'_\pm]\,dx$$
$$+\int_{-\infty}^{\infty}[lc'_\pm lC_\pm - lc_\pm lC'_\pm]\,dx + \int_{-\infty}^{\infty}[\overline{lc}'_\pm \overline{lC}_\pm - \overline{lc}_\pm \overline{lC}'_\pm]\,dx$$
$$+\frac{2\pi i\phi}{\pi - 2\gamma}\ln\left(\frac{A(\infty)}{\bar{A}(\infty)}\right). \tag{3.47}$$

Proceeding now exactly as in (2.53), introducing the dilogarithmic function and using the asymptotics as given in (3.36)–(3.39) we obtain

$$16\int_{-\infty}^{\infty} e^{-x}\,\mathrm{Re}\,lA_\pm\,dx = 2L_+\left(\omega^2(1+\omega^2)\right) + 2L_+\left(\frac{1+\omega^2}{\omega^4}\right)$$
$$+2L_+\left(\frac{\omega^4}{1+\omega^2}\right) + 2L_+\left(\frac{1}{\omega^2(1+\omega^2)}\right)$$
$$+2L_+\left(\omega^2\right) + 2L_+\left(\frac{1}{\omega^2}\right) - \frac{8\pi\phi^2}{\pi - 2\gamma}$$
$$= \pi^2 - \frac{8\pi\phi^2}{\pi - 2\gamma} \tag{3.48}$$

where we have used the functional relation (2.55) to evaluate the dilogarithms.

Hence the final result for the eigenvalue Λ is given by

$$\ln\Lambda(x - i\gamma) \simeq -Nf(x - i\gamma) - \frac{\pi i}{6N}\sinh(\pi x/\gamma)\frac{3}{2}\left[1 - \frac{8\phi^2}{\pi(\pi - 2\gamma)}\right] \tag{3.49}$$

where f denotes the bulk free energy. In this case the central charge is identified as

$$c = \frac{3}{2}\left[1 - \frac{8\phi^2}{\pi(\pi - 2\gamma)}\right] \tag{3.50}$$

for $\phi \in (-\pi/2, \pi/2)$ confirming the earlier result based on numerical solutions of the Bethe-ansatz equations for various values of γ and ϕ [28].

4. Conclusion

In conclusion, we remark that our key results are the nonlinear integral equations (2.33) and (3.34) along with the results (2.39) and (3.44) for the largest eigenvalue of the transfer matrix. The equations derived hold for finite system sizes N, from which we obtained the exact results (2.58) and (3.50) for the central charge by exploiting the symmetry properties of the kernel. Both results, for $S = \frac{1}{2}$ and $S = 1$, are in agreement with the more general formula

$$c = \frac{3S}{S+1}\left[1 - \frac{4(S+1)\phi^2}{\pi(\pi - 2S\gamma)}\right] \tag{4.1}$$

obtained from direct numerical solution of the Bethe-ansatz equations for relatively small values of S [28]. In particular, for $\phi = \gamma = \pi/(m+2S)$ this yields the minimal and superconformal series

$$c = 1 - \frac{6}{m(m+1)} \qquad c = \frac{3}{2} - \frac{12}{m(m+2)}. \tag{4.2}$$

The result (1.1) is recovered for periodic boundary conditions ($\phi = 0$). As mentioned in section 1, this result was first obtained via the known low-temperature thermodynamics [4, 18]. However, unlike the thermodynamic approach, which cannnot readily be extended to handle the excitation spectrum, we expect that the present calculations can be extended along the lines of [22] to derive the scaling dimensions, and thus the complete operator content, of the 19-vertex or Zamolodchikov–Fateev model. In fact our methods should apply to any model amenable to treatment by the previous methods using root densities [7, 10–12]. In particular, we expect that our approach can be generalized to arbitrary spin-S along the lines indicated in subsection 3.3, as well as to other boundary conditions and to models with additional surface terms [29–31].

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Appendix 1. Formulae used in the text

Here we collect some formulae which are used in the main part of the paper. We first note the Fourier transforms

$$\begin{aligned} \mathcal{F}_k\left[\ln\sinh x\right]'' &= \frac{k}{1-\mathrm{e}^{-\pi k}} \\ \mathcal{F}_k\left[\ln\coth\left(\frac{\pi x}{2\gamma}\right)\right]'' &= -\frac{k}{1+\mathrm{e}^{-\gamma k}} \end{aligned} \tag{A1.1}$$

where the variable x is assumed to lie in the upper half plane close to the real axis. For Re $a >$ Re $b > 0$ we obtain

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{bx}}{\mathrm{e}^{ax}+1}\,\mathrm{d}x = \frac{\pi}{a}\frac{1}{\sin(\pi b/a)}. \tag{A1.2}$$

Next we prove the cancellations of the symmetric kernel in our manipulations of the integral equations. This is due to the identity

$$\sum_{i,j}\int_{-\infty}^{\infty} l_i(x)(k_{ij} * l_j)'(x)\,\mathrm{d}x = \sum_{i,j}\int_{-\infty}^{\infty} l_i'(x)(k_{ij} * l_j)(x)\,\mathrm{d}x \tag{A1.3}$$

for a symmetric local kernel k_{ij} and functions l_i with constant asymptotics. In order to establish (A1.3) we perform the derivative on the left-hand side as $(k_{ij} * l_j)'(x) = (k_{ij} * l_j')(x)$, obtaining

$$\sum_{i,j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l_i(x) k_{ij}(x-y) l_j'(y) \, \mathrm{d}y \, \mathrm{d}x. \tag{A1.4}$$

Here we are allowed to interchange the order of integration. After interchanging the variables x, i with y, j and using the symmetry $k_{ji}(y-x) = k_{ij}(x-y)$ we find the right-hand side of (A1.3).

Appendix 2. 2-string deviations

In this appendix we derive the asymptotic deviation of the Bethe-ansatz numbers from the 2-string picture. For this purpose we determine the large N behaviour of the functions $\mathfrak{a}(x)$, $\mathfrak{b}(x)$ and $\mathfrak{c}(x)$. From (3.34) and $\lim_{N\to\infty} \mathfrak{A}(x) = \lim_{N\to\infty} \mathfrak{B}(x) = \lim_{N\to\infty} \mathfrak{C}(x) = 1$ we have

$$\begin{aligned} \ln \mathfrak{a}\mathfrak{c} &\simeq 2N \ln \tanh\left(\frac{\pi x}{2\gamma}\right) + \frac{2\pi \mathrm{i}\phi}{\pi - 2\gamma} \\ \ln \mathfrak{b}/\mathfrak{c} &\simeq 0 \end{aligned} \tag{A2.1}$$

These relations can be rewritten for $p(x)$ and $p(x - \mathrm{i}\gamma)$,

$$\begin{aligned} p(x) &\simeq 1 \\ p(x - \mathrm{i}\gamma) &\simeq \sqrt{2} \exp\left(-\frac{\pi \mathrm{i}\phi}{\pi - 2\gamma}\right) \left(\coth \frac{\pi x}{2\gamma}\right)^N . \end{aligned} \tag{A2.2}$$

The large N behaviour of $a(v)$ is now easily determined to be

$$\lim_{N\to\infty} a(v) = \sqrt{2} \exp\left(\frac{\pi \mathrm{i}\phi}{\pi - 2\gamma}\right) \tag{A2.3}$$

Let v_j be a Bethe-ansatz number in the lower half plane, then $v_j + \gamma\mathrm{i}$ lies in the analyticity strip of $a(v)$. Using $1/p(v_j + \gamma\mathrm{i}) = 0$ and the Bethe-ansatz condition $p(v_j) = -1$ we find

$$a(v_j + \gamma\mathrm{i}) = -\left[\tanh\left(\frac{\pi v_j}{2\gamma}\right)\right]^N . \tag{A2.4}$$

Combining the last two equations yields a convenient condition equivalent to the Bethe ansatz

$$\left[\tanh\left(\frac{\pi v_j}{2\gamma}\right)\right]^N = -\sqrt{2} \exp\left(\frac{\pi \mathrm{i}\phi}{\pi - 2\gamma}\right) . \tag{A2.5}$$

Taking logarithms we find

$$N \ln\left[\tanh\left(\frac{\pi v_j}{2\gamma}\right)\right] = (2j+1)\pi\mathrm{i} + \frac{1}{2}\ln 2 + \frac{\pi\mathrm{i}\phi}{\pi - 2\gamma} \qquad \text{(A2.6)}$$

where consecutive numbers v_j are labelled by consecutive integers $-N/2 \leqslant j < 0$. The second and the third terms on the right-hand side give rise, respectively, to a deviation from the 2-string formation and a shift of the Bethe-ansatz numbers due to the twist ϕ. Suppose both terms were absent. We would then have to deal with numbers v_j^0 subject to

$$N \ln\left[\tanh\left(\frac{\pi v_j^0}{2\gamma}\right)\right] = (2j+1)\pi\mathrm{i}. \qquad \text{(A2.7)}$$

For large N, we would then have

$$\begin{aligned} N\left\{\ln\left[\tanh\left(\frac{\pi v}{2\gamma}\right)\right]\right\}'(v_j - v_j^0) &= \frac{1}{2}\ln 2 + \frac{\pi\mathrm{i}\phi}{\pi - 2\gamma} \\ N\left\{\ln\left[\tanh\left(\frac{\pi v}{2\gamma}\right)\right]\right\}'(v_{j+1}^0 - v_j^0) &= 2\pi\mathrm{i}. \end{aligned} \qquad \text{(A2.8)}$$

In the thermodynamic limit, the Bethe-ansatz numbers are densely distributed along the line $\mathrm{Im}(v) = -\gamma/2$. Define the density function

$$\sigma(v) := \lim_{N\to\infty} \frac{1}{N(v_{j+1}^0 - v_j^0)}. \qquad \text{(A2.9)}$$

Then the deviation from the v^0 distribution is given by $\Delta v := v_j - v_j^0$, for which we find

$$N\sigma\Delta v = \frac{v_j - v_j^0}{v_{j+1}^0 - v_j^0} = -\frac{\ln 2}{4\pi}\mathrm{i} + \frac{\phi}{2(\pi - 2\gamma)}. \qquad \text{(A2.10)}$$

The deviation of the Bethe-ansatz numbers in the upper half plane is described by an analogous formula with a positive imaginary part. This is the generalization of the result obtained in [21] and [23] for periodic boundary conditions ($\phi = 0$).

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